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Research Article

Tridiagonal Operators and Zeros of Polynomials in Two Variables

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The aim of this paper is to connect the zeros of polynomials in two variables with the eigenvalues of a self-adjoint operator. This is done by use of a functional-analytic method. The polynomials in two variables are assumed to satisfy a five-term recurrence relation, similar to the three-term recurrence relation that the classical orthogonal polynomials satisfy.

1. Introduction

Orthogonal polynomials in two or more variables (also multivariate polynomials) constitute a very old subject and have been investigated by many authors using various approaches. The usefulness and applications of classical orthogonal polynomials (COP) in one variable are very well-known and thus on its own this is a very strong motivation for generalizing several results of COP to multivariate polynomials. Moreover, the potential application of multivariate orthogonal polynomials in approximation techniques and numerical methods is another strong motivation.

For example, in numerical integration, the Gauss quadrature formula

$$\int_{\alpha}^{\beta} f(x)\,\rho(x)\,dx = \sum_{i=0}^{N} w_i f(x_i), \qquad (1)$$

where x_i are the zeros of the polynomials $P_{N+1}(x)$ which are orthogonal in $[\alpha, \beta]$ with respect to $\rho(x)$, gives an approximation of the integral on the left-hand side of (1). It could be of interest to generalize (1) in two dimensions having, instead of $P_{N+1}(x)$, the two variable polynomials $P_{N+1,M+1}(x, y)$. Also, COP are used in the approximation of functions of one variable by uniquely determined series of the form

$$g(x) = \sum_i c_i P_i(x), \qquad (2)$$

where $P_i(x)$ is a sequence of COP. In a similar way, two-variable functions could be approximated by similar double series involving orthogonal bivariate polynomials $P_{i,j}(x, y)$. Moreover, an approximation of form (2) is at the "heart" of pseudospectral numerical techniques, such as the Chebyshev pseudospectral method, where $P_i(x)$ are the well-known Chebyshev polynomials $T_i(x)$. Such techniques are used for the numerical solution of one-dimensional boundary value problems and the computation of the corresponding solution is being done by evaluating the right-hand side of (2) for $P_i(x) = T_i(x)$ at specific grid points x_i which are the Gauss-Lobatto grid points. (See, e.g., [1].) It would be interesting to extend all these in the case of bivariate orthogonal polynomials satisfying a recurrence relation similar to the recurrence relation satisfied by the COP, for the straightforward numerical investigation of two-dimensional problems.

There are various extensions in the literature of the COP to polynomials of several variables or polynomials of complex variables. For example, in [2], the system $\{1, y, x, y^2, xy, x^2, \ldots\}$ was orthogonalized with respect to some region R of the xy-plane. In [3, 4], polynomials orthogonal with respect to a positive linear functional were considered. In [5], two variable analogues of classical orthogonal polynomials were constructed and studied.

The techniques used for the study of multivariate polynomials, orthogonal or not, constitute also a wide variety.