RECENT ADVANCES IN MECHANICS AND RELATED FIELDS

Special Volume
in Honour of
Professor Constantine L. Goudas

Edited by:
George Katsiaris
Vassilis Markellos
John Hadjidemetriou

Patras, 2003.

A SURVEY OF RESULTS FOR THE PAINLEVÉ EQUATIONS. THEIR APPLICATIONS AND PROPERTIES OF THEIR SOLUTIONS

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Abstract. *The purpose of this paper is to present the connection of the six non-linear ordinary Painlevé differential equations with various problems of physics, as well as with many important partial differential equations of mathematical physics. Also, some known results concerning exact, rational, meromorphic and analytic solutions of the six Painlevé equations, will be given.*

Keywords: Painlevé equations, applications, exact, rational, meromorphic, analytic solutions.

1. INTRODUCTION

Consider the well-known six Painlevé equations:

$$w''(z) = 6[w(z)]^2 + z\,, \tag{1}$$

$$w''(z) = 2[w(z)]^3 + zw(z) + \alpha\,, \tag{2}$$

$$w''(z) = \frac{[w'(z)]^2}{w(z)} - \frac{w'(z)}{z} + \frac{\alpha[w(z)]^2 + \beta}{z} + \gamma[w(z)]^3 + \frac{\delta}{w(z)}\,, \tag{3}$$

$$w''(z) = \frac{[w'(z)]^2}{2w(z)} + \frac{3}{2}[w(z)]^3 + 4z[w(z)]^2 + 2(z^2 - \alpha)w(z) + \frac{\beta}{w(z)}\,, \tag{4}$$

$$w''(z) = \left[\frac{1}{2w(z)} + \frac{1}{w(z)-1}\right][w'(z)]^2 - \frac{w'(z)}{z} + \frac{[w(z)-1]^2}{z^2}\left[\alpha w(z) + \frac{\beta}{w(z)}\right] + \frac{\gamma w(z)}{z} + \frac{\delta w(z)[w(z)+1]}{w(z)-1}\,, \tag{5}$$

$$\begin{aligned} w''(z) = {} & \frac{1}{2}\left[\frac{1}{w(z)} + \frac{1}{w(z)-1} + \frac{1}{w(z)-z}\right][w'(z)]^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w(z)-z}\right)w'(z) + \\ & + \frac{w(z)[w(z)-1][w(z)-z]}{z^2(z-1)^2} + \left[\alpha + \frac{\beta z}{(w(z))^2} + \frac{\gamma(z-1)}{(w(z)-1)^2} + \frac{\delta z(z-1)}{(w(z)-z)^2}\right], \end{aligned} \tag{6}$$

where $w = w(z)$ is, in general, a complex function of the complex variable z the primes denote differentiation with respect to z and α, β, γ and δ are arbitrary, in general complex, constants. The solutions of equations (1)-(6) define some new functions which are called the *Painlevé transcendents*. These functions are lately referred as *non-linear special functions*[1, 2], due to the fact that they have been studied a lot, like the known special functions, and they satisfy non-linear ordinary differential equations in contradiction with the known special functions, which satisfy linear ordinary differential equations.

Apart from the above mentioned Painlevé equations, there do exist their discrete analogues which are called *discrete Painlevé equations* and appear for example in quantum problems. For the derivation of these equations and various properties of them we refer for example to[3, 4]. There do exist also *matrix Painlevé equations*. Such an equation was derived in[5], were it was shown how a matrix Painlevé equation can arise in a natural way in the study of the problem of wave scattering by a broken corner. This problem is a classical problem of wave scattering subject to conditions on a boundary which consists of two rays at right angles. Because the rays do not radiate from a common point, but have a gap between them, the problem corresponds to wave scattering by a broken corner. The corresponding matrix Painlevé equation has the form[5]: