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Research Article

On the zeros of derivatives of Bessel functions

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ABSTRACT

The positive zeros of $J_\nu'''(x)$ and $J_\nu^{(n+1)}(x)$ are studied by using classical analysis and the properties of $J_\nu(x)$. It is proved that $J_\nu'''(x)$ has a unique zero in specific intervals. Regarding $J_\nu^{(n+1)}(x)$, it is proved that its positive zero $j_{\nu,m}^{(n+1)}$ is an increasing function with respect to ν, for $\nu > n$. Moreover, the first two Rayleigh sums for $j_{\nu,m}^{(n)}$ are calculated. The obtained results extend and complement previously known results and also answer an open problem regarding the monotonicity of $j_{\nu,m}^{(n)}$. As a consequence of these results, a lower bound for $j_{\nu,1}^{(n+1)}$ is deduced, as well as an inequality between $j_{\nu,1}^{(n+1)}$ and $j_{\nu,1}^{(n)}$.

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1. Introduction

The positive zeros $j_{\nu,m}$, $m = 1, 2, 3, \ldots$ of Bessel functions $J_\nu(x)$ have been extensively studied by various researchers during the last decades. Apart from the pure mathematical interest on this topic, this research was also inspired by the numerous applications of Bessel functions in applied mathematics, mathematical physics and engineering.

A specific topic of interest deals with the location and monotonicity properties of the positive zeros $j_{\nu,m}^{(n)}$, $m = 1, 2, 3, \ldots$ of Bessel function derivatives $J_\nu^{(n)}(x)$ of order $n = 1, 2, 3, \ldots$ and there are various results when $n = 1$, 2 and 3. It should be reminded that due to the symmetry with respect to 0 of the real zeros of $J_\nu^{(n)}(x)$, it suffices to study only their positive zeros. In [1,2] (see also the references therein), it was proved that the zeros $j'_{\nu,m}$ of $J'_\nu(x)$ increase with respect to ν for $\nu > -1$. In [3,4], it was proved that the zeros $j''_{\nu,m}$ of $J''_\nu(x)$ increase with respect to ν for $\nu > 0$ and an alternative proof of the monotonicity of the first zero $j''_{\nu,1}$ of $J''_\nu(x)$ for $\nu > 1$ was given in [5]. Regarding the zeros $j'''_{\nu,m}$ of $J'''_\nu(x)$ very few results can be found in the literature. More precisely, the monotonicity of $j'''_{\nu,m}$ has been studied in [6,7], whereas in [8] several theorems were proved regarding the existence of at least one zero $j'''_{\nu,m}$ in specific intervals and the monotonicity of $j'''_{\nu,1}$ for $\nu > 1$. The above-mentioned results demonstrate a pattern regarding the monotonicity of $j_{\nu,m}^{(n)}$ which was already stated in [7], but also in [9] as an open problem, i.e. '$j_{\nu,m}^{(n)}$ increase with respect to ν for $\nu > n - 1$'.

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